B.Sc. Semester-V Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 52117 **Course Code : SH/MTH/504/DSE-2**

Course Title : Boolean Algebra and Automata

OR

Probability and Statistics

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

(Boolean Algebra and Automata)

UNIT–I

1. Answer any **five** of the following questions: 2×5=10

a) Find the output sequence for AND Gate with inputs X = 111001, Y = 100101, and Z = 110011.

b) Find an equivalent regular expression of the following CFG (Context Free Grammar) $S \rightarrow aS \mid bS \mid a \mid b$.

c) Find the complement of the expression $A'B + CD'$.

d) Let $\Sigma = \{a, b\}$. Then find the language of the regular expression $(a + b)^2$.

e) Prove that any distributive lattice is modular.

f) Given Language : $L = \{ab \cup aba\}$, if X is the minimum number of states for a DFA and Y is the number of states to construct the NFA, then find $|X - Y|$.

g) Is there any difference between a finite automaton and a finite state machine? Explain.

h) Does the PCP with the following two lists have solution?

$$X = \{100, 0, 1\} \text{ and } Y = \{1, 100, 00\}.$$

Justify your answer.

UNIT–II

2. Answer any **four** of the following questions: 5×4=20

a) Show that the following grammar is ambiguous:

$$S \rightarrow aSbS \mid bSaS \mid \lambda .$$

b) What is lattice homomorphism? Give an example to show that the order relations are preserved under lattice homomorphism.

c) Simplify the following Boolean polynomials:

i) $xy + xy' + x'y$

ii) $xy' + x(yz)' + z$

d) Prove that the direct product of any two distributive lattices is a distributive lattice.

e) i) Suppose that L_1 and L_2 are two languages (over the same alphabet) given to you such that both L_1 and L_1L_2 are regular. Prove or disprove: L_2 must be regular too.

ii) Using the pumping lemma, prove that the language

$$L_3 = \{a^i b^j \mid i, j > 0, \ and \ |i-j| \ is \ a \ prime\}$$

is not regular. (Note that 1 is not treated as a prime.)

f) Prove that there exists no algorithm for deciding whether or not $L(G_1) \cap L(G_2) = \phi$ for arbitrary context-free grammars G_1 and G_2.

g) i) Give a context-free grammar for the language $L = \{a^n b^m \mid n \neq 2m\}$. Is your grammar ambiguous?

ii) Prove that $L = \{a^i b^j c^k \mid j = \max\{i, k\}\}$ is not context free.

h) Let G be a CFG in Chomsky normal form that contains b variables. Show that, if G generates some string with a derivation having at least 2b steps, L(G) is infinite.

UNIT–III

3. Answer any **one** of the following questions: 10×1=10

a) Let P be an ordered set and let $Q \subseteq P$. Show that the following are related by $(i) \Leftrightarrow (ii) \Rightarrow (iii)$ in general and are equivalent if P is a complete lattice:

i) Q is join-dense in P;

ii) $a = V_P(\downarrow a \cap Q)$ for all $a \in P$;

iii) for all $a, b \in P$ with $b < a$ there exists $x \in Q$ with $x \leq a$ and $x \not\leq b$.

b) i) Construct an NFA that accepts the following regular expression: 0 ∗ (010) ∗ (00 + 11). Then convert the NFA into equivalent DFA.

ii) Convert the grammar with following production rules to Chomsky Normal Form (CNF):

$$P = \{S \to ASB \mid \wedge, A \to aAS \mid a, B \to SbS \mid A \mid bb\}.$$

(Probability and Statistics)

UNIT–I

1. Answer any **five** of the following questions: $2\times5=10$

a) Write down the axioms of probability.

b) Let the cumulative distribution function of a random variable X is given by

$$F(x)=\begin{cases}0 & if\ \ x<-1,\\ \dfrac{x+1}{2} & if\ \ -1\le x<1,\\ 1 & if\ \ x\ge1.\end{cases}$$

What is $P\left[X>\frac{1}{2}\right]$?

c) 100 litres of water are supposed to be polluted with 10^6 bacteria. Find the probability that a sample of 1 c.c. of the same water is free from bacteria.

d) If X, Y are independent, then prove that

$$P(a<X\le b,\ c<Y\le d)=P(a<X\le b).P(c<Y\le d).$$

e) If $f(x)=\begin{cases}kx(1-x), & 0<x<1\\ 0, & \text{elsewhere}\end{cases}$ is the pdf of a random variable, then find the value of k.

f) Suppose the joint distribution of the random variables X and Y is given by

$$P(X=0,\ Y=0)=P(X=0,\ Y=1)=P(X=1,\ Y=1)=\frac{1}{3}.$$

Find out the marginal distributions of X and Y.

g) Let T_1 and T_2 be two unbiased estimators of a parameter θ. Under which condition aT_1+bT_2 will be an unbiased estimator of θ.

h) When a statistic is called consistent and unbiased estimate?

UNIT–II

2. Answer any **four** of the following questions: $5\times4=20$

a) If the joint probability density of X and Y is given by

$$f(x,y)=\begin{cases}\dfrac{2}{81}x^2y, & if\ \ 0<x<K,\ 0<y<K\\ 0, & otherwise.\end{cases}$$

i) Find the value of K so that $f(x,y)$ is a valid joint p.d.f.

ii) Find $P(X>3Y)$. 2+3

b) State and prove Chebyshev's inequality for a continuous random variable. 5

c) Show that Poisson distribution is a limiting case of the binomial distribution. 5

d) i) Find the median of X with pdf f, given by

$$f(x)=\begin{cases}1, & 0 \le x \le 1 \\ 0, & \text{otherwise}\end{cases}.$$

ii) Show that for continuous symmetrical distribution with unique median, the median and mean are equal. 2+3

e) If $\bar{X}$ be the sample mean of a random sample $(X_1, X_2, \ldots, X_n)$ drawn from an infinite population with mean μ and variance σ^2 then show that

i) $E(\bar{X})=\mu$

ii) $var(\bar{X})=\dfrac{\sigma^2}{n}$, and

iii) $E\left(\dfrac{n}{n-1}S^2\right)=\sigma^2$

where S^2 is the sample central moment of order 2. 5

f) i) If $\widehat{\Theta}_1$ is an unbiased estimator for θ, and W is a zero mean random variable, then

$$\widehat{\Theta}_2=\widehat{\Theta}_1+\mathrm{W}$$

is also an unbiased estimator for θ.

ii) If $\widehat{\Theta}_1$ is an unbiased estimator for θ such that $E\left[\widehat{\Theta}_1\right]=a\theta+b$, where $a\neq 0$, show that

$$\widehat{\Theta}_2=\frac{\widehat{\Theta}_1-b}{a}$$ is an unbiased estimator for θ.

2+3

UNIT–III

3. Answer any **one** of the following questions: 10×1=10

a) i) Let the joint probability density function of two-dimensional random variable $(X;\ Y)$ be

$$f(x,\ y)=2\left(x+y-3xy^2\right);\ 0<x<1;\ 0<y<1.$$

In this case is $E(XY)=E(X)E(Y)$?

ii) State central limit theorem for independent and identically distributed random variables with finite variance.

iii) For the two state Markov chain with transition matrix $\begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}$ and initial probability distribution (π_1, π_2), calculate the probability distribution at the n^{th} trial and show that as $n\to\infty$, this distribution is independent of the initial distribution.

4+2+4

b) i) Find the variance of Poisson distribution.

ii) Show that sample mean is an unbiased estimator of the population mean.

iii) Let X be uniformly distributed over $(0, \pi/2)$. Compute the expectation of $\sin X$. Also find the distribution of the random variable $Y = \sin X$. Show that the mean of Y is the same as the above expectation. 3+3+(2+2)

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